

GRAPH LIMITS AND HEREDITARY PROPERTIES

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ABSTRACT. We collect some general results on graph limits associated to hereditary classes of graphs. As examples, we consider some classes of intersection graphs (interval graphs, unit interval graphs, threshold graphs, chordal graphs).

1. INTRODUCTION

We use standard concepts from the theory of graph limits, see e.g. [10; 2; 3; 7]. In particular, we use the notation of [7]. Thus, \mathcal{U} is the set of all unlabelled graphs (all our graphs are finite and simple), and this is embedded (as a countable, discrete, dense, open) subset of a compact metric space $\overline{\mathcal{U}}$; the complement $\mathcal{U}_\infty := \overline{\mathcal{U}} \setminus \mathcal{U}$ is the set of graph limits, which thus itself is a compact metric space. The topology can be described by the homomorphism or subgraph numbers $t(F, G)$ or the related induced subgraph numbers $t_{\text{ind}}(F, G)$, which both extend from graphs G to general $G \in \mathcal{U}_\infty$ (see the references above for definitions, and note that F always is a graph, which we regard as fixed): A sequence of graphs (G_n) converges to a graph limit $\Gamma \iff |G_n| \rightarrow \infty$ and $t(F, G_n) \rightarrow t(F, \Gamma)$ for every graph $F \iff |G_n| \rightarrow \infty$ and $t_{\text{ind}}(F, G_n) \rightarrow t_{\text{ind}}(F, \Gamma)$ for every graph F . Moreover, a graph limit Γ is uniquely determined by the numbers $t(F, \Gamma)$ (or $t_{\text{ind}}(F, \Gamma)$) for $F \in \mathcal{U}$.

A *graph class* is a subset of the set \mathcal{U} of unlabelled graphs, i.e., a class of graphs closed under isomorphisms. Similarly, a *graph property* is a property of graphs that does not distinguish between isomorphic graphs; there is an obvious 1–1 correspondence between graph classes and graph properties and we will not distinguish between a graph property and the corresponding class. A graph class or property \mathcal{P} is *hereditary* if whenever a graph G has the property \mathcal{P} , then every induced subgraph of G also has \mathcal{P} ; this can be written

$$(1.1) \quad G \in \mathcal{P} \text{ and } t_{\text{ind}}(F, G) > 0 \implies F \in \mathcal{P}.$$

Many examples of hereditary graph classes are given in e.g. [4] and [8].

Example 1.1. Many interesting hereditary graph classes are given by various classes of intersection graphs. In general, we consider a collection \mathcal{A} of subsets of some universe and say that a graph G is an \mathcal{A} -intersection graph

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if there exists a collection of sets $\{A_i\}_{i \in V(G)} \in \mathcal{A}$ such that there is an edge $ij \in E(G)$ if and only if $A_i \cap A_j \neq \emptyset$. The class of all \mathcal{A} -intersection graphs is a hereditary graph class, for any \mathcal{A} .

Specific examples are the classes of threshold graphs and interval graphs studied in [5] and [6]. See e.g. [4] and [8] for several further examples.

Let $\mathcal{P} \subseteq \mathcal{U}$ be a graph class. We let $\overline{\mathcal{P}} \subseteq \overline{\mathcal{U}}$ be the closure of \mathcal{P} in $\overline{\mathcal{U}}$, and $\mathcal{P}_\infty := \overline{\mathcal{P}} \cap \mathcal{U}_\infty$ the set of graph limits of graphs in \mathcal{P} . Explicitly, \mathcal{P}_∞ is the set of graph limits Γ such that there exists a sequence of graphs G_n in \mathcal{P} with $G_n \rightarrow \Gamma$.

Remark 1.2. Since \mathcal{U} is open and discrete in $\overline{\mathcal{U}}$, i.e. every element of \mathcal{U} is isolated in $\overline{\mathcal{U}}$, we trivially have $\overline{\mathcal{P}} \cap \mathcal{U} = \mathcal{P}$; thus $\overline{\mathcal{P}} = \mathcal{P} \cup \mathcal{P}_\infty$. If Γ is a graph limit, then $\Gamma \in \mathcal{P}_\infty$ and $\Gamma \in \overline{\mathcal{P}}$ are equivalent, and we will use both formulations below interchangeably.

It seems to be of interest to study the classes \mathcal{P}_∞ of graph limits defined by various graph properties. Some examples have been studied, on a case-by-case basis, in [5] (threshold graphs) and [6] (interval graphs and some related graph classes), and there are many other classes that could be studied; hopefully some general pattern will emerge from the study on individual classes.

The purpose of this note is to collect a few general remarks and results; some of them from the literature and some of them new. Some further notions and facts from graph limit theory are recalled in Section 2. In Section 3 we characterize graph limits of hereditary classes of graphs using random graphs. Section 4 studies the case of classes defined by forbidding certain subgraphs, and Section 5 treats the notion of random-free graph classes, introduced by Lovász and Szegedy [12].

2. GRAPHONS AND RANDOM GRAPHS

A graph limit Γ can be represented by a *graphon*, which is a symmetric measurable function $W : \mathcal{S}^2 \rightarrow [0, 1]$ for some probability space (\mathcal{S}, μ) . Note that the representation is far from unique, see [1] and [9]. One defines, for a graph F and a graphon W ,

(2.1)

$$t(F, W) := \int_{\mathcal{S}^{|F|}} \prod_{ij \in E(F)} W(x_i, x_j) d\mu(x_1) \cdots d\mu(x_{|F|}),$$

(2.2)

$$t_{\text{ind}}(F, W) := \int_{\mathcal{S}^{|F|}} \prod_{ij \in E(F)} W(x_i, x_j) \prod_{ij \notin E(F)} (1 - W(x_i, x_j)) d\mu(x_1) \cdots d\mu(x_{|F|}),$$

and the graphon W represents the graph limit Γ that has $t(F, \Gamma) = t(F, W)$ for every graph F (or, equivalently, $t_{\text{ind}}(F, \Gamma) = t_{\text{ind}}(F, W)$ for every F).

Let X_1, X_2, \dots be an i.i.d. sequence of random elements of \mathcal{S} with distribution μ . Then (2.1)–(2.2) can be written more concisely as

$$(2.3) \quad t(F, W) = \mathbb{E} \prod_{ij \in E(F)} W(X_i, X_j),$$

$$(2.4) \quad t_{\text{ind}}(F, W) = \mathbb{E} \left(\prod_{ij \in E(F)} W(X_i, X_j) \prod_{ij \notin E(F)} (1 - W(X_i, X_j)) \right).$$

A graphon defines a random graph $G(n, W)$ with vertex set $[n] := \{1, \dots, n\}$ for every $n \geq 1$ by a standard construction: let X_1, X_2, \dots be as above, and given X_1, \dots, X_n , let ij be an edge with probability $W(X_i, X_j)$, independently for all pairs (i, j) with $1 \leq i < j \leq n$. It follows by (2.4) that if F is any graph with vertex set $[n]$, then

$$(2.5) \quad \mathbb{P}(G(n, W) = F) = t_{\text{ind}}(F, W);$$

equivalently, see (2.3),

$$(2.6) \quad \mathbb{P}(G(n, W) \supseteq F) = t(F, W).$$

This shows that the random graph $G(n, W)$ is the same (in the sense that the distribution is the same) for all graphons representing the same graph limit Γ . Thus every graph limit Γ defines a random graph $G(n, \Gamma)$ with vertex set $[n]$ for every $n \geq 1$, and this random graph can by (2.5) be defined directly by the formula

$$(2.7) \quad \mathbb{P}(G(n, \Gamma) = F) = t_{\text{ind}}(F, \Gamma)$$

for every graph F on $[n]$, which gives the distribution. As $n \rightarrow \infty$, the random graph $G(n, \Gamma)$ converges a.s. to Γ [2, Theorem 4.5].

3. GRAPH LIMITS AND RANDOM GRAPHS

Graph limits in $\overline{\mathcal{P}}$ (or equivalently, in $\mathcal{P}_\infty = \overline{\mathcal{P}} \cap \mathcal{U}_\infty$) can be characterized by the random graphs $G(n, \Gamma)$.

Theorem 3.1. *Let \mathcal{P} be a hereditary graph class and let Γ be a graph limit. Then $\Gamma \in \overline{\mathcal{P}}$ if and only if $G(n, \Gamma) \in \mathcal{P}$ a.s. for every $n \geq 1$.*

This is an immediate consequence of the following more detailed result.

Theorem 3.2. *Let \mathcal{P} be a hereditary graph class and let Γ be a graph limit. Then one of the following alternatives hold:*

- (i) $\Gamma \in \overline{\mathcal{P}}$ and $G(n, \Gamma) \in \mathcal{P}$ a.s. for every $n \geq 1$.
- (ii) $\Gamma \notin \overline{\mathcal{P}}$ and $\mathbb{P}(G(n, \Gamma) \in \mathcal{P}) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Let $\Gamma \in \overline{\mathcal{P}}$ and suppose that $G_n \rightarrow \Gamma$ with $G_n \in \mathcal{P}$. If $F \notin \mathcal{P}$, then $t_{\text{ind}}(F, G_n) = 0$ for every n by (1.1), and thus, by (2.7),

$$\mathbb{P}(G(n, \Gamma) = F) = t_{\text{ind}}(F, \Gamma) = \lim_{n \rightarrow \infty} t_{\text{ind}}(F, G_n) = 0.$$

Conversely, if $\Gamma \notin \overline{\mathcal{P}}$, then there is an open neighbourhood V of Γ in \mathcal{U}_∞ such that $V \cap \mathcal{P} = \emptyset$. As said above, $G(n, \Gamma) \rightarrow \Gamma$ a.s., which implies convergence in probability. Thus $\mathbb{P}(G(n, \Gamma) \in V) \rightarrow 1$ and $\mathbb{P}(G(n, \Gamma) \in \mathcal{P}) \leq \mathbb{P}(G(n, \Gamma) \notin V) \rightarrow 0$ as $n \rightarrow \infty$. \square

We obtain a couple of easy corollaries of Theorem 3.1.

Theorem 3.3. *Let \mathcal{P} be a hereditary graph class and let Γ be a graph limit. Then $\Gamma \in \overline{\mathcal{P}}$ if and only if $t_{\text{ind}}(F, \Gamma) = 0$ for every $F \notin \mathcal{P}$.*

Proof. Immediate by Theorem 3.1 and (2.7). \square

Theorem 3.4. *Let $\{\mathcal{P}_\alpha\}$ be a finite or infinite family of hereditary graph classes and let $\mathcal{P} = \bigcap_\alpha \mathcal{P}_\alpha$. Then $\overline{\mathcal{P}} = \bigcap_\alpha \overline{\mathcal{P}_\alpha}$.*

Thus, for example, if a graph limit is the limit of some sequence G_n of graphs in \mathcal{P}_1 , and also of another such sequence G'_n in \mathcal{P}_2 , then it is the limit of some sequence G''_n in $\mathcal{P}_1 \cap \mathcal{P}_2$. (This is not true in general, without the assumption that the classes are hereditary. For example, let $\mathcal{P}_2 = \mathcal{U} \setminus \mathcal{P}_1$, with, say, \mathcal{P}_1 the class of interval graphs.)

Proof. Suppose that $\Gamma \in \bigcap_\alpha \overline{\mathcal{P}_\alpha}$. If $F \notin \mathcal{P}$, then $F \notin \mathcal{P}_\alpha$ for some α . By assumption, $\Gamma \in \overline{\mathcal{P}_\alpha}$, and Theorem 3.3 shows that $t_{\text{ind}}(F, \Gamma) = 0$. Hence, Theorem 3.3 in the opposite direction shows that $\Gamma \in \overline{\mathcal{P}}$. The converse is obvious. (Alternatively, one can use Theorem 3.1 directly, with a little care if the family $\{\mathcal{P}_\alpha\}$ is uncountable.) \square

Remark 3.5. Conversely, we may ask whether every graph in \mathcal{P} can be obtained (with positive probability) as $G(n, \Gamma)$ for some $\Gamma \in \overline{\mathcal{P}}$ and some n . By (2.7), the class of graphs obtainable in this way equals $\bigcup_{\Gamma \in \overline{\mathcal{P}}} \mathcal{I}(\Gamma)$ where

$$(3.1) \quad \mathcal{I}(\Gamma) := \{F \in \mathcal{U} : t_{\text{ind}}(F, \Gamma) > 0\}.$$

By Theorem 3.3 (see also [12]), $\bigcup_{\Gamma \in \overline{\mathcal{P}}} \mathcal{I}(\Gamma) \subseteq \mathcal{P}$ for every hereditary graph class \mathcal{P} . Lovász and Szegedy [12] have shown that equality holds if and only if \mathcal{P} has the following property:

- (P1) *If $G \in \mathcal{P}$ and v is a vertex in G , and we enlarge G by adding a twin v' to v , i.e., a new vertex with the same neighbours as v , then at least one of the two graphs obtained by further either adding or not adding an edge vv' belongs to \mathcal{P} .*

4. FORBIDDEN SUBGRAPHS

If \mathcal{F} is a (finite or infinite) family of (unlabelled) graphs, we let $\mathcal{U}_\mathcal{F}$ be the class of all graphs that do not contain any graph from \mathcal{F} as an induced subgraph, i.e.,

$$(4.1) \quad \mathcal{U}_\mathcal{F} := \{G \in \mathcal{U} : t_{\text{ind}}(F, G) = 0 \text{ for } F \in \mathcal{F}\}.$$

This is evidently a hereditary class.

We similarly define

$$(4.2) \quad \overline{\mathcal{U}}_{\mathcal{F}} := \{\Gamma \in \overline{\mathcal{U}} : t_{\text{ind}}(F, \Gamma) = 0 \text{ for } F \in \mathcal{F}\},$$

and have the following simple result ([5, Theorem 3.2]).

Theorem 4.1. *Let $\mathcal{U}_{\mathcal{F}}$ be given by (4.1). Then $\overline{\mathcal{U}}_{\mathcal{F}} = \overline{\mathcal{U}}_{\mathcal{F}}$. In other words, if $\Gamma \in \mathcal{U}_{\infty}$ is a graph limit, then Γ is a limit of a sequence of graphs in $\mathcal{U}_{\mathcal{F}}$ if and only if $t_{\text{ind}}(F, \Gamma) = 0$ for $F \in \mathcal{F}$.*

Proof. If $G_n \rightarrow \Gamma$ with $G_n \in \mathcal{U}_{\mathcal{F}}$, then $t_{\text{ind}}(F, \Gamma) = \lim_{n \rightarrow \infty} t_{\text{ind}}(F, G_n) = 0$ for every $F \in \mathcal{F}$, by (4.1) and the continuity of $t_{\text{ind}}(F, \cdot)$. Thus $\Gamma \in \overline{\mathcal{U}}_{\mathcal{F}}$.

Conversely, suppose that $\Gamma \in \mathcal{U}_{\infty}$ and $t_{\text{ind}}(F, \Gamma) = 0$ for $F \in \mathcal{F}$. It follows from (2.7) that if $F \in \mathcal{F}$ then, for any $n \geq 1$, $G(n, \Gamma) \neq F$ a.s.; thus $G(n, \Gamma) \notin \mathcal{F}$ a.s. Moreover, every induced subgraph of $G(n, \Gamma)$ has the same distribution as $G(m, \Gamma)$ for some $m \leq n$; hence a.s. no induced subgraph belongs to \mathcal{F} and thus $G(n, \Gamma) \in \mathcal{U}_{\mathcal{F}}$. Hence $\Gamma \in \overline{\mathcal{U}}_{\mathcal{F}}$ by Theorem 3.1. \square

Remark 4.2. Every hereditary class of graphs \mathcal{P} is of the form $\mathcal{U}_{\mathcal{F}}$ for some \mathcal{F} ; we can simply take $\mathcal{F} := \mathcal{U} \setminus \mathcal{P}$. (In this case, Theorem 4.1 reduces to Theorem 3.3.) However, we are mainly interested in cases when \mathcal{F} is a small family.

Example 4.3 (Diaconis, Holmes and Janson [5]). The class \mathcal{T} of *threshold graphs* equals $\mathcal{U}_{\{2K_2, P_4, C_4\}}$. Thus, if Γ is a graph limit, then $\Gamma \in \overline{\mathcal{T}}$ if and only if $t_{\text{ind}}(P_4, \Gamma) = t_{\text{ind}}(C_4, \Gamma) = t_{\text{ind}}(2K_2, \Gamma) = 0$.

Example 4.4. The class \mathcal{UI} of *unit interval graphs* equals the class of graphs that contain no induced subgraph isomorphic to C_k for any $k \geq 4$, $K_{1,3}$, S_3 or \overline{S}_3 , where S_3 is the graph on 6 vertices $\{1, \dots, 6\}$ with edge set $\{12, 13, 23, 14, 25, 36\}$, and \overline{S}_3 is its complement [4, Theorem 7.1.9]. Thus, if Γ is a graph limit, $\Gamma \in \overline{\mathcal{UI}}$ if and only if $t_{\text{ind}}(F, \Gamma) = 0$ for every $F \in \{C_k\}_{k \geq 4} \cup \{K_{1,3}, S_3, \overline{S}_3\}$.

Example 4.5. The class \mathcal{CR} of *cographs* equals $\mathcal{U}_{\{P_4\}}$, see [4, in particular Theorem 11.3.3] where several alternative characterizations are given. Thus, if Γ is a graph limit, $\Gamma \in \overline{\mathcal{CR}}$ if and only if $t_{\text{ind}}(P_4, \Gamma) = 0$. Such graph limits are studied in Lovász and Szegedy [11].

We obtain just as easily a corresponding result for subclasses of a given hereditary graph class obtained by forbidding induced subgraphs.

Theorem 4.6. *Let \mathcal{P} be a hereditary graph class and define, for a family \mathcal{F} of graphs,*

$$(4.3) \quad \mathcal{P}_{\mathcal{F}} := \{G \in \mathcal{P} : t_{\text{ind}}(F, G) = 0 \text{ for } F \in \mathcal{F}\} = \mathcal{P} \cap \mathcal{U}_{\mathcal{F}}.$$

Then

$$(4.4) \quad \overline{\mathcal{P}_{\mathcal{F}}} = \overline{\mathcal{P}}_{\mathcal{F}} := \{\Gamma \in \overline{\mathcal{P}} : t_{\text{ind}}(F, \Gamma) = 0 \text{ for } F \in \mathcal{F}\}.$$

Proof. An immediate consequence of Theorems 4.1 and 3.4. \square

Example 4.7. The class \mathcal{UI} of *unit interval graphs* equals also the class of interval graphs that contain no induced subgraph isomorphic to $K_{1,3}$ [4, p. 111]. Thus, if \mathcal{I} is the class of all interval graphs, then $\mathcal{UI} = \mathcal{I}_{\{K_{1,3}\}}$ and hence $\overline{\mathcal{UI}} = \overline{\mathcal{I}_{\{K_{1,3}\}}}$. In other words, if Γ is a graph limit, then $\Gamma \in \overline{\mathcal{UI}}$ if and only if $\Gamma \in \overline{\mathcal{I}}$ and $t_{\text{ind}}(K_{1,3}, \Gamma) = 0$.

We have here considered induced subgraphs. We obtain similar results if we forbid general subgraphs. Let $\mathcal{U}_{\mathcal{F}}^*$ be the class of all graphs that do not contain any graph from \mathcal{F} as a subgraph, i.e.,

$$(4.5) \quad \mathcal{U}_{\mathcal{F}}^* := \{G \in \mathcal{U} : t(F, G) = 0 \text{ for } F \in \mathcal{F}\}.$$

This is evidently a hereditary class. In fact, this can be seen as a special case of forbidding induced subgraphs, since $\mathcal{U}_{\mathcal{F}}^* = \mathcal{U}_{\mathcal{F}^*}^*$, where \mathcal{F}^* is the family of all graphs H that contain a spanning subgraph F (i.e., a subgraph $F \subseteq H$ with $|F| = |H|$) with $F \in \mathcal{F}$.

Theorem 4.8. *Let $\mathcal{U}_{\mathcal{F}}^*$ be given by (4.5). Then*

$$(4.6) \quad \overline{\mathcal{U}_{\mathcal{F}}^*} = \{\Gamma \in \overline{\mathcal{U}} : t(F, \Gamma) = 0 \text{ for } F \in \mathcal{F}\}.$$

Proof. By the argument in the proof of Theorem 4.1, or by Theorem 4.1 applied to \mathcal{F}^* . \square

There is also a version corresponding to Theorem 4.6.

Example 4.9. The class of *triangle-free graphs* is $\mathcal{U}_{\{K_3\}}^* = \mathcal{U}_{\{K_3\}}^*$. Thus Theorem 4.1 and Theorem 4.8 both apply and show that the triangle free graph limits are the graph limits Γ with $t(\Gamma, K_3) = 0$. (We have $t_{\text{ind}}(K_3, \Gamma) = t(K_3, \Gamma)$ for any graph limit Γ .)

Example 4.10. The class of *bipartite graphs* equals $\mathcal{U}_{\{C_3, C_5, \dots\}}^*$. Thus a graph limit Γ is a limit of bipartite graphs if and only if $t(C_k, \Gamma) = 0$ for every odd $k \geq 3$. (We treat here bipartite graphs as a special case of simple graphs. Bipartite graphs, with an explicit bipartition, can also be treated separately, with a corresponding but distinct limit theory, see e.g. [7].)

5. RANDOM-FREE GRAPH LIMITS AND CLASSES

A graphon W is said to be *random-free* if it is 0/1-valued a.e. (In this case, the random graph $G(n, W)$ depends only on the random points X_1, X_2, \dots , without further randomization, which is a reason for the name.) If W_1 and W_2 are two graphons that represent the same graph limit, and one of them is random-free, then both are, see [9] for a detailed proof. Consequently, we can define a graph limit Γ to be random-free if some representing graphon is random-free, and in this case every representing graphon is random-free.

Random-free graphons are studied in [9, Section 9], where it is shown, for example, that a graph limit Γ is random-free if and only if the random graph $G(n, \Gamma)$ has entropy $o(n^2)$ (thus quantifying a sense in which there is less

randomness than otherwise). Another result in [9] is that Γ is random-free if and only if it is a limit of a sequence of graphs in the stronger metric δ_1 .

Lovász and Szegedy [12] define a graph property to be *random-free* if every graph limit $\Gamma \in \overline{\mathcal{P}}$ is random-free. They show some consequences of this. Moreover, they show that a hereditary graph property \mathcal{P} is random-free if and only if the following property holds:

(P2) *There exists a bipartite graph F with bipartition (V_1, V_2) such that no graph obtained from F by adding edges within V_1 and V_2 is in \mathcal{P} .*

The representation theorems in [6] show that the class of *interval graphs* is random-free, together with the classes of *circular-arc graphs* and *circle graphs*. (Of course, then every subclass is also random-free, for example the class of *threshold graphs* for which random-free graphons were found in [5].)

Remark 5.1. We do not know explicit examples of graphs F satisfying (P2) for the random-free classes of graphs just mentioned, but we guess that small examples exist.

However, not every class of intersection graphs is random-free.

Example 5.2. Let \mathcal{A} be the family of finite subsets of some infinite set (for example \mathbb{N}). Then, every graph is an \mathcal{A} -intersection graph. (If G is a graph, label the edges by integers and for each vertex v , let A_v be the set of the edges incident to v .)

Consequently, every graph limit is an \mathcal{A} -intersection graph limit, and the class of \mathcal{A} -intersection graphs is not random-free.

Example 5.3. The class of *chordal graphs* or *triangulated graphs* is the class of all graphs not containing any induced C_k with $k \geq 4$, i.e., $\mathcal{U}_{\{C_4, C_5, \dots\}}$; see [4, Section 1.2] and [8, Chapter 4] for other equivalent characterizations (and further names). We show that this class does not satisfy (P2); thus it is *not* random-free.

Let F be a bipartite graph with bipartition (V_1, V_2) , and let F_1 be the graph obtained by adding all edges inside V_1 to F . We shall show that F_1 is chordal, which shows that (P2) does not hold.

Assume that a cycle C is an induced subgraph of F_1 . If C has two vertices in V_1 , then these are adjacent in F_1 so they have to be adjacent in C . Thus, C has at most 3 vertices in V_1 ; if it has 3, then C has no other vertices, and if it has 2, they have to be adjacent in C . Hence, there are at most 2 edges in C than go between V_1 and V_2 . Since F_1 , and thus C , has no edges inside V_2 , it follows that C has at most one vertex in V_2 . Consequently, C has in every case at most 3 vertices.

We have shown that F_1 has no induced cycle of length greater than 3, i.e., F_1 is chordal. Thus no graph F as in (P2) exists, so by the result of Lovász and Szegedy [12], the class of chordal graphs is *not* random-free.

In fact, we can easily construct a graphon that defines a chordal graph limit but is not random-free. Let $\mathcal{S} = \{0, 1\}$ be a space with two points, say with measure $\frac{1}{2}$ each, and let W be the graphon $W(x, y) = (x + y)/2$ defined

on \mathcal{S} . (Alternatively, one can take $\mathcal{S} = [0, 1]$ and $W(x, y) = (\lfloor 2x \rfloor + \lfloor 2y \rfloor)/2$.) The random graph $G(n, W)$ is of the type of F_1 above (with V_1 the set of vertices i with $X_i = 1$); thus the argument above shows that $G(n, W)$ is chordal. By Theorem 3.1, the graph limit Γ_W generated by the graphon W is a chordal graph limit, and it is evidently not random-free.

Note that the class of chordal graphs is a class of intersection graphs; in fact, it is the class of intersection graphs of subtrees in a tree [8, Section 4.5]. (In order to make this fit the formulation in Example 1.1, we take an infinite universal tree T , containing all finite trees as subtrees, for example constructed by taking disjoint copies of all finite trees and joining them to a common root. We then let \mathcal{A} be the family of all finite subtrees of T .) This is thus a non-trivial class of intersection graphs that is not random-free.

We do not know any natural representation of chordal graph limits, and leave that as an open problem.

Problem 5.4. Investigate for other classes of intersection graphs whether they are random-free or not.

Example 5.5. The class \mathcal{CR} of *cographs* is random-free. This is shown in Lovász and Szegedy [11] for regular graphons in $\overline{\mathcal{CR}}$; we show the general case by verifying (P2), using the result of Lovász and Szegedy [12]. We take the graph F as the bipartite graph with 12 vertices $A, B, C, D, E, F, a, b, c, d, e, f$ and edge set $\{Ab, Bb, Cc, Dd, Ee, Ff, Ac, Bd, Ae, Bf, Ca, Cb, Db, Fa\}$.

Suppose that there exists a graph F_1 obtained by adding edges within $V_1 = \{A, B, C, D, E, F\}$ and $V_2 = \{a, b, c, d, e, f\}$ to F such that $F_1 \in \mathcal{CR}$, i.e., F_1 contains no induced P_4 . If exactly one of AB and ab is an edge in F_1 , then $aABb$ or $AabB$ is an induced P_4 ; hence either both are edges or neither is; let us write this as $AB \iff ab$. Similarly, $CD \iff cd$, $EF \iff ef$, $AB \iff cd$, $AB \iff ef$. Consequently, if AB is an edge, then so are ab and EF , and then $EFab$ is an induced P_4 ; conversely, if AB is not an edge then neither is ab nor CD and then $aCbD$ is an induced P_4 . Hence no such graph F_1 without induced P_4 exists, so (P2) holds.

Problem 5.6. Investigate for other classes of graphs with forbidden (induced) subgraphs whether they are random-free or not.

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